Comparison of Confidence Intervals for the Parameters of the Weibull and Extreme Value Distributions

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Key Words—**Weibull distribution, Extreme value distribution, *s*-Confidence interval**

Reader Aids—
Purpose: Widen state of the art
Special math needed for explanations: Statistics
Special math needed to use results: Same
Result useful to: Reliability statisticians

Summary & Conclusions—**For complete and type II censored data, this work compares the mean lengths of three exact confidence intervals for the parameters of the Weibull and extreme value distributions. The confidence intervals include conditional procedure, and unconditional method using pivotals based on the best linear invariant estimator and on the maximum likelihood estimator. The unconditional method using pivotal quantity based on the maximum likelihood estimator generally has the shortest mean length, though the three results agree fairly closely.**

1. INTRODUCTION

Point estimation for the parameters of the Weibull distribution has been extensively discussed in the literature. Gibbons & Vance [5] compared the performance of seven estimators for complete and type-II censored samples. In practice, it is more desirable to have *s*-confidence intervals than point estimates. There are two main approaches to obtain exact *s*-confidence intervals for the Weibull and extreme value distributions, namely, the conditional and unconditional; they are briefly reviewed in section 2. Some previous examples [8] showed that when the sample is not severely censored or when the sample size is large enough, the two procedures give *s*-confidence intervals which agree well. However, the operating characteristics for the two types of intervals have not been compared. In this work, a simulation study compares the mean lengths of three *s*-confidence intervals (conditional, unconditional using maximum likelihood, and best linear invariant estimators) for complete and type-II censored samples of sizes from 4 to 20. Section 2 briefly outlines the three procedures under consideration. Section 3 provides Monte Carlo simulation results. An earlier version of this paper was presented at the Statistical Conference in Taiwan, R. O. China, 1984 January.

Notation

BLIE best linear invariant estimator
MLE maximum likelihood estimator
α, β scale and shape parameters of a Weibull r.v.
u, b location and scale parameters of an extreme value r.v.
n sample size
$X_{(1)} \le X_{(2)} \le \dots X_{(r)}$ smallest r observations in a sample of size n, $r \le n$
$\hat{u}, \hat{b}$ equivariant estimators of u and b
Z_1 $(\hat{u} - u)/\hat{b}$
Z_2 $\hat{b}/b$
$\ell_1, \ell_2, \ell_1', \ell_2'$ percentage points
$\mathbf{a}$ $(a_1, a_2, \dots, a_r)$, $a_i = (X_{(i)} - \hat{u})/\hat{b}$
m_1, m_2 Cdf points of Z_1 given $\mathbf{a}$
h_2 pdf of Z_2
$\gamma(r, s)$ incomplete gamma function, $r > 0, s > 0$

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

Nomenclature

Pivotal quantity — quantity whose distribution is parameter free.
Equivariant estimator of a location (scale) parameter — estimator has the same location and scale changes (scale change) as the location and scale changes on the data.
Ancillary statistic — statistic whose distribution is parameter free.

Assumption

The data are complete or type-II censored, ie, the first r order statistics are observed, $r \le n$.

2. METHODS

The Sf of a r.v. T having a Weibull distribution is:

$$\text{weifc}(t/\alpha; \beta) \equiv \exp[-(t/\alpha)^{\beta}], \; t > 0, \tag{1}$$

where $\alpha, \beta > 0$ are parameters. It is equivalent to considering the extreme value distribution with Sf:

$$\exp[-\exp((x-u)/b)], \; -\infty < x < \infty, \tag{2}$$

since if T has Sf (1), then $X \equiv \ln T$ has Sf (2), with location parameter $u \equiv \ln\alpha$ and scale parameter $b \equiv 1/\beta$. For convenience, we discuss interval estimation of u and b in (2). Let $\hat{u}$ and $\hat{b}$ be equivariant estimators of u and b, then Lawless [7] proved that:

$$Z_1 = (\hat{u} - u)/\hat{b}, \qquad Z_2 = \hat{b}/b \tag{3}$$

are pivotal quantities. Thus s-confidence intervals of u and b based on a particular pair of equivariant estimators can be obtained. There are two different approaches: conditional and unconditional.

2.1 *Unconditional Method*

We first find the points ℓ_1, ℓ_2, ℓ_1', ℓ_2' such that:

$$\Pr\{Z_1 \le \ell_1\} = \Pr\{Z_1 \ge \ell_2\} = \alpha/2,$$

$$\Pr\{Z_2 \le \ell_1'\} = \Pr\{Z_2 \ge \ell_2'\} = \alpha/2.$$

Then $(\hat{u} - \ell_2\hat{b}, \hat{u} - \ell_1\hat{b})$ and $(\hat{b}/\ell_2', \hat{b}/\ell_1')$ are $1 - \alpha$ s-confidence intervals for u and b respectively.

The most commonly used equivariant estimators for forming Z_1 and Z_2 are the MLE and BLIE [11, 13, 17, 20]; see [8, 16] for calculating these estimates. MLEs must be calculated iteratively on a computer (the same procedure can be applied for all sample sizes). The BLIEs are easily calculated, though tables of constants are necessary for constructing estimates. The tables in [10] cover $2 \le r \le n \le 15$.

The practical difficulty for this approach is that the distributions of Z_1 and Z_2 are usually intractable, thus tables of approximate percentage points have been constructed in the literature by Monte Carlo simulation. For pivotals based on BLIEs, Mann et al. [15] gave tables for samples with $3 \le r \le n \le 25$; and for pivotals based on MLEs, the percentage points were listed in [2, 17, 18, 21] only for limited combinations of n and r.

2.2 *Conditional Method*

This approach proposed by Lawless [6, 7] is to consider the conditional distributions of Z_1 and Z_2 for given $\mathbf{a} = (a_1, a_2, \ldots, a_r)$, where $a_i = (X_{(i)} - \hat{u})/\hat{b}$ and $\hat{u}$ and $\hat{b}$ can be any pair of equivariant estimators. The quantities $a_1, a_2, \ldots, a_r$ are ancillary statistics and the conditional pdf of Z_2 given $\mathbf{a}$ is [7]:

$$h_2(z|\mathbf{a}) = \left(k(\mathbf{a}, r, n)z^{r-2} \exp\left[(z-1)\sum_{i=1}^{r} a_i\right]\right) \Big/ [f(z, \mathbf{a})/r]^r,\ z > 0,\ f(z, \mathbf{a}) \equiv \sum_{i=1}^{r} \exp(za_i) + (n-r)\exp(za_r), \tag{4}$$

where $k(\mathbf{a}, r, n)$ is the pdf normalizing constant. The conditional Cdf of Z_1 given $\mathbf{a}$ is:

$$\Pr\{Z_1 \le t|\mathbf{a}\} = \int_0^\infty h_2(z|\mathbf{a})\, \gamma(r, e^{tz}f(z, \mathbf{a}))dz$$

$$\gamma(r, s) \equiv \int_0^s u^{r-1} e^{-u} du/\Gamma(r),\ r > 0, s > 0. \tag{5}$$

Let m_1, m_2 be the percentage points of the conditional distribution of Z_1 given $\mathbf{a}$; that is,

$$\Pr\{Z_1 \le m_1|\mathbf{a}\} = \alpha/2 = \Pr\{Z_1 \ge m_2|\mathbf{a}\}.$$

Then $(\hat{u} - m_2\hat{b}, \hat{u} - m_1\hat{b})$ is a $1 - \alpha$ s-confidence interval for u. s-Confidence intervals for b can be similarly obtained.

Lawless [7] showed that for a given sample, the conditional method gives the same s-confidence intervals, regardless of what estimators are used for forming the pivotals. In the simulation comparison provided in section 3, we use the MLEs due to the convenience in programming, although any other equivariant estimators would suffice.

This method applies to all situations; it requires the use of a computer program to evaluate numerically the normalizing constant in (4), to approximate integrals in (5), and to obtain percentage points iteratively for each new sample. The details of the computational procedures are provided in [7].

3. COMPARISON

We anticipated that for large sample sizes, the three methods in section 2 would yield intervals that agree. Very accurate approximate methods are available in [1, 3, 4, 9, 12, 14] for sufficiently large samples. Thus we mainly concentrated on the numerical comparison for small to moderate sample sizes, $n = 4$ to 20. For any fixed n, several values of r representing different censoring schemes were chosen. We considered 18 combinations of n and r. For each sample, the numerical integration and iterative procedures involved in a conditional method require relatively more computing time; thus only 500 simulation samples were generated from a standard extreme value distribution ($u = 0$, $b = 1$) because of computer-time limitation. The lengths of the three 2-sided equal-tail s-confidence intervals with s-confidence level 0.90 were calculated for both parameters for each generated data set. Finally, these 500 lengths for each method were averaged to estimate the mean length. All the computations were performed on a CDC CYBER-172 computer. The results for both parameters are given in the table. In some cases the percentage points for the unconditional distributions of pivotals based on the MLEs were not given in the existing tables. In those cases, we generated 10^4 samples to provide the required approximate percentage points.

The table shows that even in very small sample sizes or in heavily censored samples, the three results are fairly close. Generally speaking, the unconditional method using pivotals based on MLEs has the shortest mean length for both parameters.

ACKNOWLEDGMENT

This research was supported by the National Science Council of the Rep. of China.

TABLE

Mean Interval Length for the Location and Scale Parameters of the Extreme Value Distribution, s-Confidence Level = 90%.

		Location Parameter			Scale Parameter		
n	r	Cond. (MLE)	Uncond. (BLIE)	Uncond. (MLE)	Cond. (MLE)	Uncond. (BLIE)	Uncond. (MLE)
4	4	2.36*	2.43	2.38	2.31*	2.34	2.31*
	3	3.92*	4.00	4.04	4.26*	4.46	4.26*
5	5	2.01	1.99*	2.01	1.85	1.84*	1.84*
	3	4.49	4.56	4.40*	4.48	4.62	4.31*
8	8	1.38	1.44	1.37*	1.17	1.18	1.15*
	6	1.75	1.74*	1.74*	1.68*	1.70	1.68*
	4	3.15	3.12	3.04*	2.89	2.92	2.65*
	3	6.28	6.07*	6.36	4.72	4.74	4.64*
10	10	1.20*	1.25	1.24	.99*	1.04	1.03
	8	1.41*	1.44	1.41*	1.27*	1.28	1.27*
	5	2.46	2.46	2.37*	2.12	2.14	2.07*
	3	7.77	7.78	7.65*	5.14	5.07	5.05*
15	15	0.95*	0.95*	0.95*	0.76	0.76	0.75*
	10	1.24*	1.30	1.25	1.14	1.16	1.13*
20	20	0.82*	0.83	0.82*	0.65	0.63*	0.64
	15	0.93*	0.94	0.93*	0.84	0.85	0.83*
	10	1.39	1.39	1.37*	1.17	1.19	1.16*
	5	4.07	3.89*	4.13	2.42	2.31*	2.44

* denotes the shortest among three lengths

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Manuscript TR84-063 received 1984 July 18; revised 1985 October 27.

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