Approximate Mean Squared Errors of Estimators of Reliability for k-out-of-m Systems in the Independent Exponential Case

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Approximate Mean Squared Errors of Estimators of Reliability for *k*-out-of-*m* Systems in the Independent Exponential Case

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Suppose S is a system with m independent components such that the system functions if and only if at least k components operate. Assume that the component lifetimes are identically exponentially distributed. This work provides simple approximation formulas for the mean squared errors of the minimum variance unbiased and the maximum likelihood estimators of reliability of this system. The performance of the proposed approximation formulas is investigated.

KEY WORDS: Reliability; k-out-of-m system; Minimum variance unbiased estimator; Maximum likelihood estimator.

1. INTRODUCTION

Suppose S is a k-out-of-m system. That is, S is a system with m components such that the system functions if and only if at least k components operate successfully. When $k = m$ and $k = 1$, we obtain series and parallel systems as special cases. It is assumed that the component lifetimes are independent and that each follows the exponential distribution

$$f(x, \theta) = \theta^{-1} \exp(-x/\theta), \quad 0 < \theta < \infty, \quad 0 < x < \infty. \quad (1.1)$$

The parameter θ is unknown. Let n_i replicates of the ith component be separately tested, $n = \sum_1^m n_i$. Then the lifetimes of these n tested components, $X_1, X_2, \ldots, X_n$, are a random sample from distribution (1.1). If $k = m = 1$, the system is known as a one-unit system. A multicomponent device whose system lifetime follows the same distribution as (1.1) can be regarded as a one-unit system. In this case, the observations are made on the performance of the whole system, operating as one complex unit. The reliability of the system at a mission time t, $R(t)$, is the probability that the system will function successfully for at least t time units. Under the assumed life distribution, Pugh (1963) and Basu (1964) obtained the minimum variance unbiased estimator (MVUE) of the reliability for one-unit systems. Rutemiller (1966) investigated series and parallel cases. The general construction of the MVUE and the maximum likelihood estimator (MLE) for k-out-of-m systems was treated by Basu and El Mawaziny (1978), who indicated that the estimators are asymptotically equivalent. Hence for large sample sizes, either the MVUE or the MLE can be used. The question for moderate and small sample situations is which estimator should be employed. Zacks and Even (1966a) studied the small sample relative efficiency of both estimators for one-unit systems. One common measure of relative efficiency is the ratio of mean squared errors. The mean squared errors given in Zacks and Even (1966a, b), however, are not easily calculable, since they are given in terms of exponential integrals or modified Bessel functions. Basu and El Mawaziny (1978) used Monte Carlo simulation to find the mean squared errors for 1-out-of-3, 2-out-of-3, and 3-out-of-3 systems. In this study, we obtain simple approximation formulas for the mean squared errors of both estimators.

We will discuss one-unit systems in Section 2 and k-out-of-m systems in Section 3. Numerical results and comparisons are made in the final section. Although we deal exclusively with the exponential distribution in this study, we believe that the method of approximation used here will prove useful in approximating other incomplete integrals arising in statistical applications.

2. ONE-UNIT SYSTEM

Let X be the lifetime of a one-unit system. X is assumed to have density function (1.1). Thus the reliability of this system at time t is

$$R(t) = P(X > t) = \exp(-\lambda),$$

where $\lambda = t/\theta$. Pugh (1963) and Basu (1964) proved that the MVUE of $R(t)$ is

$$\hat{R}(t) = [1 - (t/T)]^{n-1} \quad \text{if } t < T, \qquad (2.1)$$
$$= 0 \quad \text{if } t \geq T,$$

where $T = \sum_{i=1}^n X_i$. It is well known that T has the density function

$$f(v) = [\theta^n \Gamma(n)]^{-1} v^{n-1} \exp(-v/\theta), \quad 0 < v < \infty.$$

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Hence from (2.1), we have

$$E[\hat{R}(t)]^2 = \int_t^\infty [1 - (t/u)]^{2n-2}[\theta^n\Gamma(n)]^{-1} \times u^{n-1}\exp(-u/\theta)du. \tag{2.2}$$

If we let $v = u - t$ in (2.2), it follows that

$$\begin{aligned} E[\hat{R}(t)]^2 &= \exp(-\lambda)\int_0^\infty [1 + (t/v)]^{1-n} \\ &\quad \times [\theta^n\Gamma(n)]^{-1}v^{n-1}\exp(-v/\theta)dv \qquad (2.3) \\ &= \exp(-\lambda)\,E\{\exp[(1 - n)\ln(1 + t/n\bar{X})]\}, \end{aligned}$$

where $\bar{X}$ denotes the sample mean.

We now will determine the approximation formula in an asymptotic way. After expanding $\ln(1 + t/n\bar{X})$ and collecting terms by the order of n, we can write

$$\begin{aligned} &E\{\exp[(1 - n)\ln(1 + t/n\bar{X})]\} \\ &= E[\exp(-t/\bar{X})] \\ &\quad + n^{-1}E\left[\exp(-t/\bar{X})\left(\frac{t}{\bar{X}} + \frac{t^2}{2\bar{X}^2}\right)\right] \qquad (2.4) \\ &\quad + n^{-2}E\left[\exp(-t/\bar{X})\left(\frac{t^3}{6\bar{X}^3} + \frac{t^4}{8\bar{X}^4}\right)\right] + 0(n^{-3}). \end{aligned}$$

We now examine (2.4) term by term. Note that

$$\exp(-t/\bar{X}) = R(t) + \sum_{i=1}^{5} R^{(i)}(\theta)(\bar{X} - \theta)^i/i! + R_5, \tag{2.5}$$

where $R^{(i)}$ is the ith derivative of $R(t)$ and

$$R_5 = R^{(6)}(u)(\bar{X} - \theta)^6/6!,$$

where u is a number between $\bar{X}$ and θ. The following moments are needed.

$$\begin{aligned} E(\bar{X} - \theta)^2 &= n^{-1}\theta^2, \\ E(\bar{X} - \theta)^3 &= n^{-2}(2\theta^3), \\ E(\bar{X} - \theta)^4 &= n^{-2}(3\theta^4) + n^{-3}(6\theta^4), \qquad (2.6) \\ E(\bar{X} - \theta)^5 &= n^{-3}(20\theta^5) + n^{-4}(24\theta^5), \\ E(\bar{X} - \theta)^6 &= n^{-3}(15\theta^6)\,n^{-4}(130\theta^6) + n^{-5}(120\theta^6). \end{aligned}$$

It is easy to verify that $R^{(6)}(t)$ is bounded for $0 < t < \infty$; thus $ER_5 = O(n^{-4})$. Using (2.6), we have after some manipulations

$$\begin{aligned} &E[\exp(-t/\bar{X})] - R(t) \\ &= \exp(-\lambda)\left[n^{-1}\left(-\lambda + \frac{\lambda^2}{2}\right)\right. \\ &\quad \left. + n^{-2}(-\lambda + \tfrac{5}{2}\lambda^2 - \tfrac{7}{6}\lambda^2 + \tfrac{1}{8}\lambda^4)\right] + 0(n^{-3}). \end{aligned} \tag{2.7}$$

Similar procedures can be applied to evaluate the other terms of (2.4). The following results can be shown.

$$\begin{aligned} &E\left[\exp(-t/\bar{X})\left(\frac{t}{\bar{X}} + \frac{t^2}{2\bar{X}^2}\right)\right] \\ &= \exp(-\lambda)\left[\left(\lambda + \frac{\lambda^2}{2}\right)\right. \\ &\quad \left. + n^{-1}\left(\lambda - \frac{\lambda^2}{2} - \lambda^3 + \frac{\lambda^4}{4}\right)\right] + O(n^{-2}), \qquad (2.8) \\ &E\left[\exp(-t/\bar{X})\left(\frac{t^3}{6\bar{X}^3} + \frac{t^4}{8\bar{X}^4}\right)\right] \\ &= \exp(-\lambda)\left(\frac{\lambda^3}{6} + \frac{\lambda^4}{8}\right) + O(n^{-1}). \end{aligned}$$

Substituting (2.8) and (2.7) into (2.4), we then have

$$\begin{aligned} E[\hat{R}(t)]^2 &= \exp(-2\lambda)\left[1 + n^{-1}\lambda^2\right. \\ &\quad \left. + n^{-2}\lambda^2\left(2 - 2\lambda + \frac{\lambda^2}{2}\right)\right] + O(n^{-3}). \end{aligned}$$

Consequently, the asymptotic mean squared error of the MVUE is

$$\begin{aligned} &E[\hat{R}(t)]^2 - [R(t)]^2 = \exp(-2\lambda)\lambda^2 \\ &\quad \times \left[n^{-1} + n^{-2}\left(2 - 2\lambda + \frac{\lambda^2}{2}\right)\right] + O(n^{-3}). \qquad (2.9) \end{aligned}$$

We now proceed to find the approximate mean squared error of the MLE. The MLE of $R(t)$ is

$$\tilde{R}(t) = \exp(-t/\bar{X}).$$

The asymptotic bias of the MLE is essentially given in (2.7). As for the mean squared error, note that from (2.5),

$$\begin{aligned} &E[\tilde{R}(t) - R(t)]^2 \\ &= [R^{(1)}(\theta)]^2E(\bar{X} - \theta)^2 \\ &\quad + [R^{(1)}(\theta)R^{(2)}(\theta)]E(\bar{X} - \theta)^3 \\ &\quad + \left[\frac{[R^{(2)}(\theta)]^2}{4} + \frac{R^{(1)}(\theta)R^{(3)}(\theta)}{3}\right]E(\bar{X} - \theta)^4 \\ &\quad + O(n^{-3}). \end{aligned}$$

Easy substitution leads to the following expression for the asymptotic mean squared error of the MLE:

$$\exp(-2\lambda)\lambda^2[n^{-1} + n^{-2} \times (5 - 7\lambda + \tfrac{7}{4}\lambda^2)] + O(n^{-3}). \tag{2.10}$$

3. *k*-OUT-OF-*m* SYSTEM

In the independent exponential case, the reliability of a k-out-of-m system is

$$\begin{aligned} R(t) &= \sum_{j=k}^{m}\binom{m}{j}(1 - F(t))^j(F(t))^{m-j} \\ &= \sum_{j=k}^{m}\sum_{\alpha=0}^{m-j}(-1)^\alpha\binom{m}{j}\binom{m-j}{\alpha} \\ &\quad \times \exp[-t(j + \alpha)/\theta]. \end{aligned}$$

The MVUE given in Basu and El Mawaziny (1978) has the following form:

$$\hat{R}(t) = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} (-1)^{\alpha} \binom{m}{j} \binom{m-j}{\alpha} \times \left[\left(1 - \frac{t(j+\alpha)}{T} \right)^{+} \right]^{n-1}, \tag{3.1}$$

where

$$(1 - t(j + \alpha)/T)^{+} = \max\,[0, 1 - t(j + \alpha)/T].$$

The MLE is given by

$$\tilde{R}(t) = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} (-1)^{\alpha} \binom{m}{j} \binom{m-j}{\alpha} \times \exp\,[-t(j + \alpha)/\bar{X}].$$

Note that

$$\begin{aligned} E[\tilde{R}(t) - R(t)]^2 &= \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} \sum_{l=k}^{m} \sum_{\beta=0}^{m-l} (-1)^{\alpha+\beta} \times \binom{m}{j}\binom{m}{l}\binom{m-j}{\sigma}\binom{m-l}{\beta} \\ &\quad \times \{E[\exp(-t(j+\alpha+l+\beta)/\bar{X})] \\ &\quad - \exp(-\lambda(l+\beta))E[\exp(-t(j+\alpha)/\bar{X})] \\ &\quad - \exp(-\lambda(j+\alpha))E[\exp(-t(l+\beta)/\bar{X})] \\ &\quad + \exp(-\lambda(j+\alpha+l+\beta))\}. \end{aligned}$$

Applying (2.7) to each term involving expectation, we can obtain the following approximate mean squared error.

$$\begin{aligned} &\sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} \sum_{l=k}^{m} \sum_{\beta=0}^{m-l} (-1)^{\alpha+\beta} \binom{m}{j}\binom{m}{l} \\ &\times \binom{m-j}{\alpha}\binom{m-l}{\beta} \exp(-\lambda(j+\alpha+l+\beta))\lambda^2 \\ &\times (j+\alpha)(l+\beta)\{n^{-1} + n^{-2}[5 - \tfrac{7}{2}(j+\alpha)\lambda \\ &- \tfrac{7}{2}(l+\beta)\lambda + \tfrac{1}{2}(j+\alpha^2\lambda^2 + \tfrac{1}{2}(l+\beta)^2\lambda^2 \\ &+ \tfrac{3}{4}(j+\alpha)(l+\beta)\lambda^2]\} + 0(n^{-3}). \end{aligned} \tag{3.2}$$

As a special case, if $k = m = 1$, (3.2) reduces to (2.10).

From (3.1), it follows that

$$E[\hat{R}(t)]^2 = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} \sum_{l=k}^{m} \sum_{\beta=0}^{m-l} (-1)^{\alpha+\beta} \times \binom{m}{j}\binom{m}{l}\binom{m-j}{\alpha}\binom{m-l}{\beta} \cdot I,$$

where

$$I = \int_{w}^{\infty} \left[1 - \frac{(j+\alpha+l+\beta)t}{u} + \frac{(j+\alpha)(l+\beta)t^2}{u^2} \right]^{n-1} \times [\theta^n \Gamma(n)]^{-1} \exp(-u/\theta) u^{n-1} du$$

and $w = \max\,[(j + \alpha)t, (l + \beta)t]$. Similar steps to those used in obtaining (2.3) can be applied to the integral I. We can write

$$\begin{aligned} I &= \exp(-w/\theta)E \times \left[1 + \frac{w}{n\bar{X}} - \frac{(j+\alpha+l+\beta)t}{n\bar{X}} + \frac{(j+\alpha)(l+\beta)t^2}{n^2\bar{X}^2 + nw\bar{X}} \right]^{n-1} \\ &= \exp(-w/\theta)E \times \frac{[w - (j+\alpha)t][w - (l+\beta)t] + n^2\bar{X}^2 + 2nw\bar{X} - (j+\alpha+l+\beta)tn\bar{X}}{n^2\bar{X}^2 + nw\bar{X}} \\ &= \exp(-w/\theta)E \times \left[\frac{n\bar{X} + 2w - (j+\alpha+l+\beta)t}{n\bar{X} + w} \right]^{n-1}, \end{aligned}$$

since $[w - (j + \alpha)t]\,[w - (l + \beta)t] = 0$. Note further that

$$\begin{aligned} I = \exp(-w/\theta)E &\times \left\{ \exp\left[(n-1)\ln\left(1 + \frac{2w - (j+\alpha+l+\beta)t}{n\bar{X}}\right) \right.\right. \\ &\left.\left. - (n-1)\ln\left(1 + \frac{w}{n\bar{X}}\right) \right] \right\}. \end{aligned}$$

When we let either $w = (j + \alpha)t$ or $w = (l + \beta)t$ in the preceding formula and use the same asymptotic method as in the previous section, both cases lead to the same result given by

$$\begin{aligned} E[\hat{R}(t)]^2 - [R(t)]^2 &= \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} \sum_{l=k}^{m} \sum_{\beta=0}^{m-l} (-1)^{\alpha+\beta} \times \binom{m}{j}\binom{m}{l}\binom{m-j}{\alpha}\binom{m-l}{\beta} \\ &\quad \cdot \exp(-\lambda(j+\alpha+l+\beta))(j+\alpha)(l+\beta)\lambda^2 \\ &\quad + \{n^{-1} + n^{-2}[2 - (j+\alpha)\lambda - (l+\beta)\lambda \\ &\quad + \tfrac{1}{2}(j+\alpha)(l+\beta)\lambda^2]\} + O(n^{-3}). \end{aligned} \tag{3.3}$$

Note that if $k = m = 1$, (3.3) reduces to (2.9).

Similar approximation results can be obtained if the sampling information is censored. If we assume that the first $r(0 < r \leq n)$ ordered observations $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ of a complete sample of size n are available, the MLE of $R(t)$ is

$$\tilde{R}(t) = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} (-1)^{\alpha} \binom{m}{j}\binom{m-j}{\alpha} \times \exp[-t(j+\alpha)/U],$$

where U is the MLE of θ in the case of censored data. That is,

$$U = \left[\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)} \right] \Big/ r.$$

The MVUE obtained in Basu and El Mawaziny (1978) has the following form:

$$\hat{R}(t) = \sum_{j=k}^{m} \sum_{\alpha \Lambda 0}^{m-j} (-1)^{\alpha} \binom{m}{j} \binom{m-j}{\alpha} \times \left[\left(1 - \frac{(j+\alpha)t}{V} \right)^{+} \right]^{r-1},$$

where $V = rU$. It is well known that

$$V = \sum_{i=1}^{r} (n - i + 1)(X_{(i)} - X_{(i-1)}), \quad (X_{(0)} = 0),$$

and $(n - i + 1)(X_{(i)} - X_{(i-1)}, i = 1, 2, \ldots, r$ are independently and identically distributed and have the density (1.1). Thus $\tilde{R}(t)$ and $\hat{R}(t)$ can be considered individually as the MLE and MVUE of $R(t)$ from a complete sample of size r chosen from distribution (1.1). Consequently, if n in (3.2) and (3.3) is replaced by r, we then get the approximate MSE's of the MLE and MVUE, respectively, for the censored case.

4. NUMERICAL RESULTS AND COMPARISONS

4.1 One-Unit System

In this section, we investigate the performance of the proposed approximation formulas (2.9) and (2.10). The approximate results will be compared with the exact values (formulas given in Zacks and Even 1966a) for several sample sizes. Since the estimates are between zero and one, all the mean squared errors are relatively small. The relative error will thus be considered for comparisons. For each fixed sample size, we choose 201 different λ's; that is, $\lambda = 0.01, 0.03, 0.05, \ldots, 4.01$. Then for each specific λ, the relative error is calculated. Finally, these 201 relative errors (corresponding to 201 λ's) are averaged. In Table 1, we list the average relative errors for sample sizes 4 to 16.

As we expected, the error decreases when the sample size becomes large. We can see the approximation formulas provided in this study are satisfactory even in small sample sizes, although they are derived in an asymptotic way.

By comparing the mean squared errors of both estimators specifically for sample sizes 4 and 8, Zacks and Even (1966a) found that when λ is between roughly .5 and 3.5, the MLE is more efficient than the MVUE. If we employ our approximation formulas (2.9) and (2.10), we can easily see that when λ is between .7 and 3.3, the MLE has smaller mean squared error for any sample size $n \geq 5$ (for which the approximate results are regarded satisfactory). Otherwise, the MVUE is more efficient.

Table 1. Average Relative Error (%)

n	MLE	MVUE
4	17.37	3.21
5	11.99	2.21
6	8.80	1.62
7	6.76	1.23
8	5.36	.97
9	4.35	.78
10	3.61	.65
11	3.05	.52
12	2.60	.46
13	2.25	.40
14	1.97	.35
15	1.73	.30
16	1.52	.27

Table 2. Approximate Intervals Where the MLE Has Smaller MSE by Using (3.2) and (3.3)

k	m	Interval
1	1	(.7, 3.3)
1	2	(1.1, 3.5)
2	2	(.4, 1.6)
1	3	(1.3, 3.7)
2	3	(.6, 1.9)
3	3	(.3, 1.1)
1	4	(1.4, 3.8)
2	4	(.8, 2.1)
3	4	(.4, 1.3)
4	4	(.2, .8)
1	5	(1.6, 3.9)
2	5	(.9, 2.3)
3	5	(.6, 1.5)
4	5	(.3, 1.0)
5	5	(.2, .7)

4.2 *k*-out-of-*m* System

Two-unit systems connected in parallel or in series are studied in Zacks and Even (1966b). Based on the numerical results particularly for $n = 8$, they conclude for 2-out-of-2 systems that when λ is between roughly .25 and 2.00, the MLE is more efficient than the MVUE. In Basu and El Mawaziny (1978), Monte Carlo estimates show that neither is uniformly better than the other for i-out-of-3 systems ($i = 1, 2, 3$). Approximation formulas (3.2) and (3.3) can be easily employed to find the interval where the MLE has smaller mean squared error. Note that from (3.2) and (3.3), the approximate MSE's for the MLE and MVUE have the form $An^{-1} + Bn^{-2} + O(n^{-3})$ and $An^{-1} + Cn^{-2} + O(n^{-3})$, respectively. Hence if we ignore $O(n^{-3})$ term, the interval for which $An^{-1} + Bn^{-2} < An^{-1} + Cn^{-2}$ is independent of n. Such intervals are tabulated in Table 2 for $m = 1$ to 5 based on our numerical printouts. Those intervals are valid for any n for which the formulas (3.2) and (3.3) are considered to be satisfactory.

For each λ, according to Table 2, we can predict which estimator is better. Those predictions agree with the findings of Monte Carlo estimates for $n = 10$ in Basu and El Mawaziny (1978).

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