On Comparing Estimators of $Pr\{Y < X\}$ in the Exponential Case

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Key Words—**Exponential distribution, Maximum likelihood estimator, Minimum variance s-unbiased estimator.**

Reader Aids—
Purpose: Report a derivation
Special math needed for explanations: Statistics
Special math needed to use results: Same
Results useful to: Statisticians, Reliability theoreticians

Abstract—**Let X and Y be s-independent exponentially distributed random variables with mean β and α respectively. This work povides simple approximations for s-bias and mean square error of the maximum likelihood estimator of $\Pr\{Y < X\}$ for two cases: 1) both α and β are unknown; 2) only β is unknown. When α is known, the mean square error is compared with that of the minimum variance s-unbiased estimator and a preference relationship between them is established using the mean square error criterion.**

1. INTRODUCTION

The problem of estimating $Pr\{Y < X\}$ has been extensively studied in reliability and related fields. It arises when a component of strength X is subjected to a stress Y. The component fails when and only when $X \leqslant Y$. If X and Y are s-independently distributed as exponential random variables with mean β and α respectively, we have reliability

$$R = Pr\{Y < X\} = \beta/(\alpha + \beta). \tag{1a}$$

The exponential case has been discussed in [1, 4, 5, 7 - 9]. We first assume that one of the parameters, say α, is known and, without losing generality, let $\alpha = 1$. In this case,

$$R = \beta/(1 + \beta). \tag{1b}$$

Suppose $X_1, X_2, \ldots, X_n$ is a random sample chosen from distribution of X. The minimum variance s-unbiased estimator (MVUE) $\hat{R}$ of R is [8]:

$$\hat{R} = \frac{(n-1)!(-1)^{n-1}}{(n\bar{X})^{n-1}} \left[\sum_{i=0}^{n-1} \frac{(-1)^i (n\bar{X})^i}{i!} - e^{-n\bar{X}} \right], \tag{2}$$

$$\bar{X} \equiv \sum_{i=1}^{n} X_i/n.$$

Using a result in [2], [5] showed that the mean square error (MSE) of $\hat{R}$ is:

$$\text{MSE}\{\hat{R}\} = \frac{1}{(1+\beta)^2} \sum_{k=1}^{\infty} \left(\frac{\beta}{1+\beta}\right)^{2k} \Big/ \binom{n+k-1}{k} \tag{3a}$$

$$= \frac{\beta^2}{(1+\beta)^4} n^{-1} + \frac{2\beta^4}{(1+\beta)^6} n^{-2} + \frac{-2\beta^4 - 4\beta^5 + 4\beta^6}{(1+\beta)^8} n^{-3} + \frac{2\beta^4 + 8\beta^5 - 6\beta^6 - 28\beta^7 + 8\beta^8}{(1+\beta)^{10}} n^{-4} + O(n^{-5}). \tag{3b}$$

The maximum likelihood estimator (MLE) when $\alpha = 1$ is:

$$\tilde{R} = \bar{X}/(1 + \bar{X}). \tag{4a}$$

Both estimators were compared [5] using 15-point Gaussian Laguerre quadrature to obtain approximate MSE and s-bias of the MLE. For large n, the precision of this quadrature was not sufficient for accurate calculation of the first two moments of the MLE. Therefore, it seems worthwhile to provide simple and satisfactory approximation formulas for s-bias and MSE of the MLE, although bounds were obtained [7]. Section 2 derives such formulas. The MLE and MVUE are also compared.

If both α and β are unknown, and $Y_1, Y_2, \ldots, Y_n$ is a random sample from Y, then the MLE of R is:

$$\tilde{R} = \bar{X}/(\bar{X} + \bar{Y}), \quad \bar{Y} \equiv \sum_{i=1}^{n} Y_i/n. \tag{4b}$$

Ref. [7] also developed bounds for s-bias of $\tilde{R}$ in this case. However, the suggested bounds did not have explicit expressions. Section 3 presents corresponding suitable approximations.

2. APPROXIMATIONS WHEN $\alpha = 1$

Let $f(x) \equiv x/(1 + x)$.

$$\tilde{R} - R = f(\bar{X}) - f(\beta)$$

$$= \sum_{i=1}^{8} f^{(i)}(\beta)(\bar{X} - \beta)^i/i! + r_8, \tag{5}$$

$f^{(i)}(x) \equiv (-1)^{i+1} i!(1 + x)^{-(i+1)}$, i-th derivative of $f(x)$,

$r_8 \equiv f^{(9)}(u)(\bar{X} - \beta)^9/9!$, is a remainder,

u is a number between $\bar{X}$ and β. The central moments of $\bar{X}$ are:

$$E\{\overline{X} - \beta\} = 0,$$

$$E\{(\overline{X} - \beta)^2\} = \beta^2 n^{-1},$$

$$E\{(\overline{X} - \beta)^3\} = (2\beta^3)n^{-2},$$

$$E\{(\overline{X} - \beta^4)\} = (3\beta^4)n^{-2} + (6\beta^4)n^{-3}, \qquad (6)$$

$$E\{(\overline{X} - \beta)^5\} = (20\beta^5)n^{-3} + (24\beta^5)n^{-4},$$

$$E\{(\overline{X} - \beta)^6\} = (15\beta^6)n^{-3} + (130\beta^6)n^{-4} + (120\beta^6)n^{-5},$$

$$E\{(\overline{X} - \beta)^7\} = (210\beta^7)n^{-4} + (924\beta^7)n^{-5} + (720\beta^7)n^{-6},$$

$$E\{(\overline{X} - \beta)^8\} = (105\beta^8)n^{-4} + (2380\beta^8)n^{-5} + (7308\beta^8)n^{-6} + (5040\beta^8)n^{-7}.$$

$E\{r_8\} = O(n^{-5})$. Take s-expectation on both sides of (5) and substitute the preceding moments.

$$E\{\tilde{R}\} - R = \frac{-\beta^2}{(1+\beta)^3} n^{-1} + \frac{2\beta^3 - \beta^4}{(1+\beta)^5} n^{-2} + \frac{-6\beta^4 + 8\beta^5 - \beta^6}{(1+\beta)^7} n^{-3} + \frac{24\beta^5 - 58\beta^6 + 22\beta^7 - \beta^8}{(1+\beta)^9} n^{-4} + O(n^{-5}). \qquad (7)$$

We proceed to find the approximate MSE. From (5), we have

$$E\{(\tilde{R} - R)^2\} = [f^{(1)}]^2 E\{(\overline{X} - \beta)^2\} + [f^{(1)} f^{(2)}] \cdot E\{(\overline{X} - \beta)^3\} + \left\{\frac{[f^{(2)}]^2}{4} + \frac{f^{(1)} f^{(3)}}{3}\right\} E\{(\overline{X} - \beta)^4\} + \left\{\frac{f^{(1)} f^{(4)}}{12} + \frac{f^{(2)} f^{(3)}}{6}\right\} E\{(\overline{X} - \beta)^5\} + \left\{\frac{[f^{(3)}]^2}{36} + \frac{f^{(1)} f^{(5)}}{60} + \frac{f^{(2)} f^{(4)}}{24}\right\} \cdot E\{(\overline{X} - \beta)^6\} + \left\{\frac{f^{(3)} f^{(6)}}{360} + \frac{f^{(2)} f^{(5)}}{120} + \frac{f^{(3)} f^{(4)}}{72}\right\} \cdot E\{(\overline{X} - \beta)^7\} + \left\{\frac{[f^{(4)}]^2}{576} + \frac{f^{(1)} f^{(7)}}{2520} + \frac{f^{(2)} f^{(6)}}{720} + \frac{f^{(3)} f^{(5)}}{360}\right\} E\{(\overline{X} - \beta)^8\} + O(n^{-5}), \qquad (8)$$

$$f^{(i)} \equiv f^{(i)}(\beta).$$

The asymptotic MSE of the MLE is:

$$\text{MSE}\{\tilde{R}\} = \frac{\beta^2}{(1+\beta)^4} n^{-1} + \frac{-4\beta^3 + 5\beta^4}{(1+\beta)^6} n^{-2} + \frac{18\beta^4 - 44\beta^5 + 13\beta^6}{(1+\beta)^8} n^{-3} + \frac{-96\beta^5 + 362\beta^6 - 248\beta^7 + 29\beta^8}{(1+\beta)^{10}} n^{-4} + O(n^{-5}). \qquad (9)$$

If the $O(n^{-5})$ terms in (7) and (9) are ignored, we then obtain approximate s-bias and MSE of $\tilde{R}$. The results are now compared with those provided in [5], in which 15-point quadrature was used to approximate the first two moments of $\tilde{R}$. Tables 1 and 2 compare s-bias and s-efficiency respectively.

TABLE 1.
Negative s-bias of $\tilde{R}$ using (7).

R	.5	.75	.90	.95	.99
β	1.0	3.0	9.0	19.0	99.0
n = 5	.237E-1	.288E-1	.183E-1	.107E-1	.242E-2
10	.122E-1	.143E-1	.862E-2	.490E-2	.108E-2
25	.495E-2	.566E-2	.332E-2	.186E-2	.408E-3
50	.249E-2	.282E-2	.164E-2	.917E-3	.200E-3
100	.125E-2	.141E-2	.815E-3	.455E-3	.990E-4
max dif.	0	0	0	1	0

E implies an exponent to base 10.
max dif. is the maximum difference with results of [5] in the third significant digit.

The results in tables 1 and 2 are very close to those of [5]. As anticipated, the proposed formulas are better when the sample size becomes large. Even for small sample sizes, the formulas still perform satisfactorily although the results are derived asymptotically. For large n, computational difficulty arises in calculating the 15-point Gauss-Laguerre quadrature [5] and for $R \leqslant 0.5$, $n \geqslant 50$, the quadrature is not accurate at all. Apparently, the proposed formulas do not have those drawbacks. Numerical results further show that the values of (7) are precisely between bounds obtained in [7, p 41].

A similar method has also been applied [3] to obtain the MSEs of estimators of reliability for k-out-of-m systems.

We now compare both estimators based on MSE criterion. Using (3b) and (9), we can find for any fixed n

TABLE 2
Mean square *s*-efficiency of $\hat{R}$ relative to $\tilde{R}$ ($\text{MSE}\{\tilde{R}\}/\text{MSE}\{\hat{R}\}$) using (3b) and (9).

R	.5	.75	.90	.95	.99
β	1.0	3.0	9.0	19.0	99.0
$n = 5$	.936	1.111	1.379	1.520	1.657
10	.971	1.075	1.201	1.260	1.314
25	.989	1.035	1.082	1.102	1.120
50	.995	1.018	1.041	1.051	1.059
100	.997	1.009	1.021	1.025	1.029
max dif.	0	4	12	3	105

Other than $n = 5$, the max dif is 26.
max dif. is the maximum difference with results of [5] in the third significant digit.

the approximate interval of β (or of R) for which the MLE has smaller MSE. Such intervals are given in table 3 for several sample sizes. In other words, the mean square *s*-efficiency of $\hat{R}$ relative to $\tilde{R}$ ($\text{MSE}\{\tilde{R}\}/\text{MSE}\{\hat{R}\}) \leqslant 1$ if β (or R) is located in the interval as listed. Otherwise, the MVUE is more *s*-efficient.

TABLE 3
Intervals where the MLE has smaller MSE using (3b) and (9)

n	interval of β	interval of R
5	(0, 1.67)	(0, .625)
10	(0, 1.48)	(0, .597)
15	(0, 1.43)	(0, .588)
20	(0, 1.40)	(0, .583)
30	(0, 1.37)	(0, .578)
40	(0, 1.36)	(0, .576)
50	(0, 1.36)	(0, .576)
100	(0, 1.34)	(0, .573)

As n becomes large, we can ignore the n^{-3}, n^{-4} terms in (3b) and (9). Thus it is obvious that the MLE has smaller MSE i.f.f.—

$$-4\beta^3 + 5\beta^4 \leqslant 2\beta^4, \text{ or}$$

$$\beta \leqslant 4/3, \text{ or } R \leqslant 4/7.$$

That is, the intervals of β in table 3 will converge to (0, 4/3) as n increases. The results are generally consistent with [5].

From (3b), (7), (9), we can show that

$$n^{1/2}[\hat{R} - \tilde{R}] = O(n^{-1/2}) \to 0,$$

$$\text{Var}\{n^{1/2}(\hat{R} - \tilde{R})\} = O(n^{-1}) \to 0,$$

which implies [6, p 153] that $n^{1/2}(\hat{R} - \tilde{R}) \xrightarrow{P} 0$. Hence we have established the asymptotic equivalence of both estimators.

An advantage of the proposed approximation method (besides the simplicity) is that we can easily get more accurate results just by retaining more terms in the expansion procedures.

3. APPROXIMATIONS WHEN α IS UNKNOWN

The method introduced in the previous section can be extended in an obvious way: Let $g(x, y) \equiv x/(x + y)$,

$$\tilde{R} - R = g(\overline{X}, \overline{Y}) - g(\beta, \alpha)$$

$$= \sum_{i=1}^{8} \sum_{j=0}^{i} \left(\frac{\partial^i g(x, y)}{\partial x^j \partial y^{i-j}} \right)_{x=\beta,\, y=\alpha} \cdot \frac{(\overline{X} - \beta)^j (\overline{Y} - \alpha)^{i-j}}{j!(i - j)!} + z_8, \tag{10}$$

$$z_8 \equiv \sum_{j=0}^{9} \left(\frac{\partial^9 g(x, y)}{\partial x^j \partial y^{9-j}} \right)_{x=w,\, y=v} \frac{(\overline{X} - \beta)^9 (\overline{Y} - \alpha)^{9-j}}{9!(9 - j)!},$$

w is between $\overline{X}$ and β, v is between $\overline{Y}$ and α. Using the moments of $\overline{X}$, $\overline{Y}$ and applying similar method as before, we have asymptotic *s*-bias:

$$E\{\tilde{R}\} - R = \frac{\varrho(\varrho - 1)}{(1 + \varrho)^3} \left(n^{-1} + \frac{\varrho^2 - 4\varrho + 1}{(1 + \varrho)^2} n^{-2} + \frac{\varrho^4 - 14\varrho^3 + 30\varrho^2 - 14\varrho + 1}{(1 + \varrho)^4} n^{-3} + \frac{\varrho^6 - 36\varrho^5 + 207\varrho^4 - 352\varrho^3 + 207\varrho^2 - 36\varrho + 1}{(1 + \varrho)^6} n^{-4} \right) + O(n^{-5}) \tag{11}$$

$\varrho \equiv \alpha/\beta.$

Ref. [7, p 43] shows that the *s*-bias and MSE of $\tilde{R}$ satisfy—

$$\text{MSE}\{\tilde{R}\} = \text{Bias}\{\tilde{R}\}\left[\frac{n(1 + \varrho)}{1 - \varrho} - \frac{1 - \varrho}{1 + \varrho}\right] + \frac{\varrho}{(1 + \varrho)^2}. \tag{12}$$

Consequently,

$$\text{MSE}\{\tilde{R}\} = \frac{2\varrho^2}{(1 + \varrho)^4} n^{-1} + \frac{4\varrho^2(2\varrho - 1)(\varrho - 2)}{(1 + \varrho)^6} n^{-2} + \frac{12\varrho^2(2\varrho^4 - 17\varrho^3 + 32\varrho^2 - 17\varrho + 2)}{(1 + \varrho)^8} n^{-3} + O(n^{-4}). \tag{13}$$

We now examine the performance of (11). Table 4 lists the negative *s*-bias for several values of n and ϱ. It is sufficient to give values only for $0 < \varrho < 1$ [7] which calculated bounds of *s*-bias for some sample sizes $\geqslant 10$. Almost all values are between the bounds provided in [7], and (11) is quite satisfactory even for small n.

TABLE 4
Negative *s*-Bias of R using (11).

n	5	10	25	50	100
ϱ = .01	.236E-2	.106E-2	.399E-3	.196E-3	.970E-4
.1	.148E-1	.709E-2	.276E-2	.137E-2	.680E-3
.2	.189E-1	.938E-2	.373E-2	.186E-2	.927E-3
.3	.187E-1	.947E-2	.381E-2	.191E-2	.955E-3
.4	.166E-1	.854E-2	.347E-2	.174E-2	.873E-3
.5	.138E-1	.716E-2	.292E-2	.147E-2	.738E-3
.6	.108E-1	.563E-2	.231E-2	.116E-2	.584E-3
.7	.783E-2	.409E-2	.168E-2	.847E-3	.426E-3
.8	.500E-2	.262E-2	.108E-2	.543E-3	.273E-3
.9	.239E-2	.125E-2	.515E-3	.260E-3	.131E-3

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REFERENCES

[1] A.M. Beg, "Estimation of *Pr*{$Y < X$} for exponential family," *IEEE Trans. Reliability*, vol R-29, 1980, pp 158-159.

[2] B.J.N. Blight, P.V. Rao, "The convergence of Bhattacharyya bounds," *Biometrika*, vol 61, 1974, pp 137-142.

[3] A. Chao, "Approximate mean squared errors of estimators of reliability for *k*-out-of-*m* systems in the independent exponential case," *J. Amer. Statist. Assoc.*, vol 76, 1981, pp 720-724.

[4] N.L. Johnson, "Letter to the Editor," *Technometrics*, vol 17, 1975, p 393.

[5] G.D. Kelley, J.A. Kelley, W.R. Schucany, "Efficient estimation of *Pr*{$Y < X$} in the exponential case," *Technometrics*, vol 18, 1976, pp 359-360.

[6] C.R. Rao, *Linear Statistical Inference and Its Applications*, second edition, John Wiley, New York, 1973.

[7] Y.S. Sathe, S.P. Shah, "On estimating *P*{$X > Y$} for the exponential distribution," *Comm. Statist. A*, vol A10(1), 1981, pp 39-47.

[8] H. Tong, "A note on the estimation of *Pr*{$Y < X$} in the exponential case," *Technometrics*, vol 16, 1974, p 625, Errata, vol 17, 1975, p 395.

[9] H. Tong, "On the estimation of *Pr*{$Y \leqslant X$} for exponential families," *IEEE Trans. Reliability*, vol R-26, 1977, pp 54-56.

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