Confidence Intervals for Reliability From Stress-Strength Relationships

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Abstract—**A new distribution-free procedure obtains *s*-confidence intervals for the reliability in the stress-strength model. Based on the coverage probability and average-length criteria, a simulation study compared the procedure with other methods. Generally the proposed intervals perform best in maintaining nominal coverage probabilities.**

1. INTRODUCTION

In reliability analysis from stress-strength relationships, a component with random strength is subjected to a random stress. The reliability of the component is measured by the probability that the strength exceeds the stress. This type of reliability model applies to many physical situations [2-4]. On the basis of two *s*-independent random samples from the distributions of stress and strength respectively, a distribution-free point estimate of the reliability can be obtained by using the Wilcoxon-Mann-Whitney statistic [3]. This statistic is also the minimum variance unbiased estimator of the reliability.

From practical considerations, it is desirable to have not only the point estimate of reliability but also the *s*-confidence intervals for it. In this paper, our major concern is to suggest a trustable *s*-confidence interval for the component reliability. While the setting of our problem is simple, we hope that this paper leads to better understanding of the important problem of constructing *s*-confidence intervals for more complicated system reliabilities. In fact, even for this simple stress-strength model the problem of choosing the most useful *s*-confidence interval for the reliability is far from settled.

Several *s*-confidence intervals were proposed by Birnbaum & McCarty [3], Sen [10], Govindarajulu [6], and Ury [12], among others. In section 2, we briefly discuss their merits and suggest new procedures. Section 3 provides Monte Carlo simulation results for the *s*-confidence intervals under consideration. The *s*-confidence intervals based on the *t* distribution approximation and the jackknife estimate of the variance of the reliability estimate is recommended for applications.

Notation

X	$(X_1, X_2, \dots, X_m)$, a random sample of size m from stress distribution with Cdf F.
Y	$(Y_1, Y_2, \dots, Y_n)$, a random sample of size n from strength distribution with Cdf G.
$\phi(X_i, Y_j)$	1, if $X_i < Y_j$; 0, otherwise.
F_m, G_n	empirical Cdf's of X, Y
R	$T(F, G) \equiv \int_{-\infty}^{\infty} F(x)dG(x)$, reliability
$\hat{R}$	$T(F_m, G_n) \equiv \int_{-\infty}^{\infty} F_m(x)dG_n(x)$, Wilcoxon-Mann-Whitney statistic
$F_{m-1,i}$	empirical Cdf of $(X_1, X_2, \dots, X_{i-1}, X_{i+1}, \dots, X_m)$
$G_{n-1,j}$	empirical Cdf of $(Y_1, Y_2, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_n)$
$J_1(i)$	$mT(F_m, G_n) - (m-1)T(F_{m-1,i}, G_n)$, $i = 1, 2, \dots, m$
$J_2(j)$	$nT(F_m, G_n) - (n-1)T(F_m, G_{n-1,j})$, $j = 1, 2, \dots, n$
w	$[\min\{m, n\}]^{-1/2}$
N	$m + n$
$1-\alpha$	*s*-confidence coefficient
df	degree of freedom
Z_α	α quantile of the *s*-normal distribution; $\text{gauf}(Z_\alpha) \equiv \alpha$
$t_{\nu,\alpha}$	α quantile of the Student t distribution with df, ν; $\text{stuf}(t_{\nu,\alpha}; \nu) \equiv \alpha$

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

Nomenclature

coverage probability: probability that *s*-confidence interval contains R
empirical coverage probability: proportion of the time (out of 1000 trials) that *s*-confidence intervals cover R
average length: average of interval lengths of *s*-confidence intervals for 1000 trials

2. *s*-CONFIDENCE INTERVALS FOR R

1. Birnbaum & McCarty bounds [3]

A distribution-free upper *s*-confidence bound for R is $\hat{R} + \delta/\sqrt{N}$, where δ satisfies:

$$1-\alpha = 1 - mN^{-1} e^{-2n\delta^2/N} - nN^{-1} e^{-2m\delta^2/N} - 2mnN^{-2}\delta\, e^{-2mn\delta^2/N^2} \int_{-2n\delta/N}^{2m\delta/N} e^{-t^2/2}\, dt.$$

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Values of δ as a function of sample size and s-confidence coefficient were tabulated in [3, 9]. 2-Sided intervals can be similarly obtained. However, these bounds are very crude and the results apply only when the sample sizes are large [6, p 229].

2. Sen bounds [10]

Sen showed that for $0 < R < 1$, and $w^{-2} \to \infty$, the asymptotic distribution of the variable $(mn/N)^{1/2}(\hat{R} - R)/S$ is standard s-normal, where

$$S^2 \equiv \left\{\frac{n}{m-1}\sum_{i=1}^{m}\left[\frac{1}{n}\sum_{j=1}^{n}\phi(X_i, Y_j) - \hat{R}\right]^2 + \frac{m}{n-1}\sum_{j=1}^{n}\left[\frac{1}{m}\sum_{i=1}^{m}\phi(X_i, Y_j) - \hat{R}\right]^2\right\}/N,$$

is a s-consistent estimator of $\text{Var}\{(mn/N)^{1/2}(\hat{R} - R)\}$. Therefore, an approximate $(1 - \alpha)$ s-confidence interval for R is:

$$I_1 = [\hat{R} + s\,Z_{\alpha/2}, \hat{R} + s\,Z_{1-\alpha/2}],\ s^2 \equiv NS^2/(mn) \quad (1)$$

Sen proposed another variance estimator [10, p 98]. Under some situations (for example, $m = n = 2, x_1 < y_1 < x_2 < y_2$), the variance estimates could be negative. Hence we will not consider it in our simulation study.

3. Govindarajulu bounds [6]

Govindarajulu suggested two s-confidence bounds:

$$I_2' = [\hat{R} + \tfrac{1}{2}\,w\,Z_{\alpha/2}, \hat{R} + \tfrac{1}{2}\,w\,Z_{1-\alpha/2}],$$

$$I_2 = [\hat{R} + w\,\hat{\sigma}Z_{\alpha/2}, \hat{R} + w\,\hat{\sigma}Z_{1-\alpha/2}],$$

$$\hat{\sigma}^2 \equiv \frac{w^{-2}}{n}\left[\int_{-\infty}^{\infty} F_m^2(x)dG_n(x) - \hat{R}^2\right] + \frac{w^{-2}}{m}\left[\int_{-\infty}^{\infty} G_n^2(x)dF_m(x) - (1-\hat{R})^2\right]. \quad (2)$$

$\hat{\sigma}^2$ is an unbiased, s-consistent estimator of $\text{Var}\{\hat{R}/w\}$. Our untabulated simulation results showed that the interval I_2 is, in general, shorter than I_2', and the empirical coverage probabilities of I_2 are much closer to the anticipated $1 - \alpha$ value.

4. Ury bounds [12]

Ury showed that if $1 - \alpha \le 0.925$ and the sample sizes are not too unequal, the s-confidence interval:

$$I_3 = [\hat{R} - \tfrac{1}{2}\,w\alpha^{-1/2}, \hat{R} + \tfrac{1}{2}\,w\alpha^{-1/2}] \quad (3)$$

are useful for small samples and are shorter than the one derived by Birnbaum & McCarty [3].

5. Percentile method [5]

Based on the bootstrap procedure, Efron introduced the following percentile method for obtaining s-confidence intervals:

a. Construct F_m and G_n.

b. Draw bootstrap random samples X^*, Y^*, s-independently from F_m and G_n, respectively. Compute the bootstrap replication $\hat{R}^*$, the Wilcoxon-Mann-Whitney statistic evaluated for the bootstrap samples.

c. Repeat step-b B times and obtain B s-independent bootstrap replications $\hat{R}^{*1}, \hat{R}^{*2}, \ldots, \hat{R}^{*B}$.

d. Let $H(t) \equiv$ (number of $\hat{R}^{*k} \le t$, $k = 1, \ldots, B)/B$, $H^{-1}(\alpha) \equiv \inf\{t: H(t) \ge \alpha\}$, $0 < \alpha < 1$. The percentile method gives:

$$I_4 = \left[H^{-1}\left(\frac{\alpha}{2}\right), H^{-1}\left(1 - \frac{\alpha}{2}\right)\right] \quad (4)$$

as an approximation $(1 - \alpha)$ s-confidence interval for R.

Efron also discussed the bias-corrected percentile method (pp 82-84). Our simulation study showed that the improvement from this method is quite limited for our problem. Hence we do not report the simulation results of this method.

6. Our proposal

We use the following jackknife estimator of $\text{Var}\{\hat{R}\}$:

$$V^2 \equiv \frac{1}{m(m-1)}\sum_{i=1}^{m}[J_1(i) - \bar{J}_1]^2 + \frac{1}{n(n-1)}\sum_{j=1}^{n}[J_2(j) - \bar{J}_2]^2.$$

The idea for using V^2 is given in the appendix. There are two approximations:

a. $(\hat{R} - R)/V$ is distributed as a Student t r.v. with $df = \nu$; ν is the integer part of:

$$\left\{V^4\Big/\left[\frac{\Sigma_1^m(J_1(i) - \bar{J}_1)^2}{(m-1)^2(m+1)m^2} + \frac{\Sigma_1^n(J_2(j) - \bar{J}_2)^2}{(n-1)^2(n+1)n^2}\right]\right\} - 2.$$

[8, p 25]. Thus, a $(1 - \alpha)$ s-confidence interval for R is:

$$I_5 = [\hat{R} - Vt_{\nu,1-\alpha/2}, \hat{R} + Vt_{\nu,1-\alpha/2}] \quad (5)$$

b. $(\hat{R} - R)/V$ is distributed as a Student t r.v. with $df = \varkappa$:

$$\varkappa = \left[\frac{C^2}{m-1} + \frac{1 - C^2}{n-1}\right]^{-1},$$

$$C \equiv \left[\sum_{1}^{m}(J_1(i) - \bar{J}_1)^2\right]^2 / [m(m-1)^2 V^2].$$

[1, p 219; or 13]. Thus a (1 - α) s-confidence interval for R is:

$$I_6 = [\hat{R} - Vt_{\varkappa,1-\alpha/2}, \hat{R} + Vt_{\varkappa,1-\alpha/2}]. \qquad (6)$$

The quantile is obtained by linear interpolation when $\varkappa$ is not an integer.

3. MONTE CARLO STUDY

In order to understand better the performance of these s-confidence intervals, we consider some simulated results under various sample sizes. The simulations involve exponential and Weibull distributions:

Table 1

X: exponential with Sf$\{x\}$ = exp($-x/2$),
Y: exponential with Sf$\{y\}$ = exp($-y/3$),
R = 0.600.

Table 2

X: Exponential with Sf$\{x\}$ = exp($-2x$),
Y: Weibull with Sf$\{y\}$ = exp($-y^2$),
R = 0.7584.

Table 3

X: Weibull with Sf$\{x\}$ = exp($-x^3$),
Y: Weibull with Sf$\{y\}$ = exp($-y^2/4$),
R = .8069.

Ref [7, chapter 6] shows calculation formulas for R. In each case, 1000 replications of two s-independent samples X and Y were generated and sample sizes ($m = n = 5$), ($m = 5, n = 10$), ($m = 10, n = 5$), ($m = n = 10$), ($m = n = 20$) were used. For the percentile method, the number of bootstrap replications for each generated sample was B = 1000. Tables 1-3 provide empirical coverage probabilities and average of the interval lengths of the s-confidence intervals I_j, j = 1, 2, ..., 6 at s-confidence coefficient = 0.90.

In general, a desirable s-confidence interval should have short interval length with coverage probability close enough to the anticipated s-confidence coefficient.

These tables show that I_3, suggested by Ury has the longest (worst) average length and the empirical coverage probabilities for all cases are 100% (much too high). This indicates that I_3 is too long. The intervals I_4 derived from the percentile method perform badly when sample sizes are small. Govindarajulu's interval I_2, and Sen's interval I_1 are slightly better but still not satisfactory, because the coverage probabilities are not close enough to the anticipated 90%. Intervals I_5 and I_6 are quite competitive. However, for all cases, I_5 is shorter than I_6. Using the above desirability criterion, we conclude that interval I_5, based on the jackknife estimate of the variance and the t distribution approximation, performs best.

TABLE 1
Empirical Coverage Probability and Average of Interval Length When X and Y Have Exponential Distributions, $1 - \alpha = 0.90$. I_j, j = 1, 2, ..., 6 are given in (1)-(6).

m	n	criterion	I_1	I_2	I_3^*	I_4	I_5	I_6
5	5	coverage	.860	.853	1.0	.835	.909	.919
		length	.638	.571	1.0	.580	.749	.828
5	10	coverage	.868	.838	1.0	.856	.893	.904
		length	.522	.480	1.0	.493	.585	.617
10	5	coverage	.847	.814	1.0	.827	.880	.909
		length	.546	.498	1.0	.512	.630	.694
10	10	coverage	.886	.865	1.0	.887	.903	.917
		length	.434	.412	1.0	.421	.462	.484
20	20	coverage	.883	.875	1.0	.885	.884	.902
		length	.300	.292	.707	.296	.301	.316

*We use max$\{1, w\alpha^{-1/2}\}$ as the length of I_3.

TABLE 2
Empirical Coverage Probability and Average of Interval Length When X Has an Exponential Distribution and Y Has a Weibull Distribution, $1 - \alpha = 0.90$.

m	n	criterion	I_1	I_2	I_3^*	I_4	I_5	I_6
5	5	coverage	.840	.830	1.0	.774	.933	.938
		length	.568	.508	1.0	.480	.677	.737
5	10	coverage	.820	.803	1.0	.798	.856	.859
		length	.476	.432	1.0	.426	.565	.584
10	5	coverage	.872	.853	1.0	.865	.902	.912
		length	.429	.397	1.0	.399	.475	.528
10	10	coverage	.869	.857	1.0	.859	.880	.893
		length	.369	.350	1.0	.356	.395	.411
20	20	coverage	.883	.875	1.0	.880	.888	.901
		length	.258	.251	.707	.254	.261	.272

TABLE 3
Empirical Coverage Probability and Average of Interval Length When X and Y Have Weibull Distributions, $1 - \alpha = 0.90$.

m	n	criterion	I_1	I_2	I_3^*	I_4	I_5	I_6
5	5	coverage	.885	.868	1.0	.721	.899	.904
		length	.540	.483	1.0	.406	.656	.701
5	10	coverage	.894	.845	1.0	.846	.899	.924
		length	.383	.357	1.0	.347	.418	.435
10	5	coverage	.818	.778	1.0	.596	.826	.861
		length	.472	.426	1.0	.133	.583	.609
10	10	coverage	.873	.854	1.0	.863	.893	.899
		length	.348	.331	1.0	.330	.379	.389
20	20	coverage	.884	.874	1.0	.873	.893	.898
		length	.240	.234	.707	.236	.248	.253

APPENDIX
Idea for using V^2

Following Serfling [11, section 6.6.1] we define the influence curves of the estimator $T(F_m, G_n)$ for $T(F, G)$ by:

$$h_1(F, G; x) \equiv \frac{dT(F + \epsilon(\delta_x - F), G + \gamma(\delta_y - G))}{d\epsilon}\Bigg|_{\substack{\epsilon=0,\\ \gamma=0}}$$

$$h_2(F, G; y) \equiv \frac{dT(F + \epsilon(\delta_x - F), G + \gamma(\delta_y - G))}{d\gamma}\Bigg|_{\substack{\epsilon=0,\\ \gamma=0}}$$

where δ_c is the Cdf degenerate at c. It can be shown that:

$$T(F_m, G_n) - T(F, G) = \frac{1}{m} \sum_{i=1}^{m} h_1(F, G; X_i) + \frac{1}{n} \sum_{j=1}^{n} h_2(F, G; Y_j) + R_{m,n}, \qquad \text{(A1)}$$

where $R_{m,n} = o_p\left(\left(\frac{N}{mn}\right)^{1/2}\right)$, $m, n \to \infty$.

Then from (A1), we have:

$$J_1(i) \approx h_1(F, G; X_i) + \frac{1}{n} \sum_{j=1}^{n} h_2(F, G; Y_j) + T(F, G),$$

$$J_2(j) \approx \frac{1}{m} \sum_{i=1}^{m} h_1(F, G; X_i) + h_2(F, G; Y_j) + T(F, G).$$

Since $R_{m,n}$ in (A1) is negligible, the estimator of Var$\{\hat{R}\}$ should be —

$$V_0^2 \equiv \frac{1}{m(m-1)} \sum_{i=1}^{m} [h_1(F, G; X_i) - \bar{h}_1]^2 + \frac{1}{n(n-1)} \sum_{j=1}^{n} [h_2(F, G; Y_j) - \bar{h}_2]^2$$

$$\approx \frac{1}{m(m-1)} \sum_{i=1}^{m} [J_1(i) - \bar{J}_1]^2 + \frac{1}{n(n-1)} \sum_{j=1}^{n} [J_2(j) - \bar{J}_2]^2 = V^2,$$

This suggests V^2 can be used as an estimator of Var$\{\hat{R}\}$. It is not difficult to show that V^2 is exactly the same as the variance estimator proposed by Sen [10]. This further justifies the use of the estimator V^2.

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